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Space–time topology and spontaneous symmetry breaking

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Abstract. Spontaneous symmetry breaking of a gauge theory with topological charge is studied in an arbitrary space–time. Sets of relations on the topological charges are found which, if not satisfied, inhibit the symmetry breaking from that particular sector. Different solutions to these relations correspond to a symmetry breaking into topologically inequivalent gauge sectors of the residual symmetry group.

1. Introduction

Many physical applications of gauge theories invoke spontaneous symmetry breaking in order that various particles may acquire masses without disturbing the renormalisability of the quantised theory. Especially important examples are quantum chromodynamics, Salam–Weinberg weak/electromagnetic unification and grand unified theories of the weak, electromagnetic and strong interactions. Currently there is much interest in the role played by these theories in the early stages of the universe. However, in such a regime, space–time structure may differ significantly from the conventional, flat, Minkowskian picture and it becomes important to understand how, for example, a non-trivial topology affects the symmetry breaking mechanism. Topologically complex space–times also occur in the Riemannian approach to quantum gravity (Hawking 1979, Pope 1981 for a recent review) and topological ideas have found many applications in studies of quantum field theory in a fixed, unquantised, gravitational background. (A bibliography and brief review are contained in Isham (1981a).) Non-trivial manifolds also arise in conventional instanton theory if unusual boundary conditions are employed ('t Hooft 1979) or in magnetic monopole theory if the region containing the monopoles is excised.

There have been various investigations of spontaneous symmetry breaking in a curved space–time, and the role of space–time topology is discussed in Avis and Isham (1978, 1979), Banach (1980), Birrell (1981), Ford (1980), Toms (1980, 1981). All these papers are concerned with a scalar field ϕ and the Z_2 action $\phi \to -\phi$. This problem is of special interest when the space–time M is not simply connected and non-trivial real line bundles may occur. The cross sections of such bundles are 'twisted' scalar fields (Isham 1978) and necessarily vanish somewhere. In particular the conventional solution, in which ϕ equals a non-zero constant, does not exist and in this sense the Z_2 symmetry breaking is inhibited.

Non-trivial line bundles are classified by elements of the cohomology group $H^1(M; Z_2) \approx \mathrm{Hom}(\pi_1(M), Z_2)$ and in the present paper an extensive generalisation is presented in which the initial and final symmetry groups are an arbitrary Lie group G and subgroup H respectively. The aim is to study a G-gauge theory in an arbitrary manifold M and to find necessary and sufficient conditions for the occurrence of symmetry breaking from G to H (written $G \Rightarrow H$). We shall see that these involve the 'topological charges' of the initial G-sector. For certain pairs (G, H) there are always some charge values which allow $G \Rightarrow H$, whilst for other pairs the ability to satisfy these conditions depends on fine details of the space–time topology. An additional possibility is the breaking of a given G-sector into a number of inequivalent H-sectors. This splitting phenomenon will be denoted $G \lesseqgtr H$.

Most of these effects only occur on a manifold of dimension four (or above), and for this reason attention will be focused on a compact four-dimensional space–time. Thus the investigation lies within the framework of Riemannian quantum foam (Hawking 1978) and gravitational instantons, rather than within the canonical quantisation schemes frequently employed in discussions of twisted scalar fields.

The topological results were announced in Isham (1981b), but with no proofs and a sketched method which involved a Postnikov decomposition of a certain fibration. A simpler technique is employed in the present paper which sidesteps the Postnikov study by invoking the fact that all G-bundles over a manifold of dimension four (or less) may be cohomologically classified by sets of characteristic classes. The proof of this result involves Postnikov techniques and in this sense the two approaches are equivalent. A number of other results were described in the preliminary paper, including a demonstration of the existence of a classical phase change in vacuum solutions to the field equations as the coupling constant is varied. These results will not be duplicated here and I shall concentrate on the topological aspects of the problem.

Section 2 contains a short review of the fibre bundle picture of spontaneous symmetry breaking in a general space–time. The solution of the $G \Rightarrow H$ and $G \lesseqgtr H$ problems is discussed in § 3 using the theory of universal bundles and characteristic classes. A number of specific examples are displayed in § 4, including the pairs of groups (G, H) most commonly employed in grand unified theories.

2. Spontaneous symmetry breaking in a curved space–time

2.1.

A standard Lagrangian contains a gauge field $\boldsymbol{\omega}_\mu$ and a G-multiplet $\boldsymbol{\phi}$ of scalar fields. The field strength $\boldsymbol{F}_{\mu\nu}$ is defined as

$$F^i_{\mu\nu} = \partial_\nu \omega^i_\mu - \partial_\mu \omega^i_\nu + C^i_{jk} \omega^j_\nu \omega^k_\mu \tag{2.1}$$

where C^i_{jk} are the structure constants of the symmetry group G. The theory is defined on a compact four-manifold equipped with a Riemannian metric $g_{\mu\nu}$ and has the action

$$S = \int_m \left(\tfrac{1}{4} g^{\mu\alpha} g^{\nu\beta} \boldsymbol{F}_{\mu\nu} \cdot \boldsymbol{F}_{\alpha\beta} + \tfrac{1}{2} \mathscr{D}_\mu \boldsymbol{\phi} \cdot \mathscr{D}^\mu \boldsymbol{\phi} + V(\boldsymbol{\phi})\right) \sqrt{g}\, \mathrm{d}^4 x. \tag{2.2}$$

The field equations are

$$\mathscr{D}_\mu \boldsymbol{F}^{\mu\nu} + \phi^i \boldsymbol{T}_{ij} \mathscr{D}^\nu \phi^j = 0, \tag{2.3}$$

$$\mathscr{D}_\mu \mathscr{D}^\mu \phi^i - \partial V / \partial \phi^i = 0, \tag{2.4}$$

where $\boldsymbol{T}_{ij}$ is the Lie algebra representation matrix arising in the covariant derivative $\mathscr{D}_\mu \phi^i = \partial_\mu \phi^i + \boldsymbol{\omega}_\mu \cdot \boldsymbol{T}^i{}_j \phi^j$. It is to be understood that the covariant derivatives in (2.3) and (2.4) include Christoffel symbol contributions where appropriate.

The solutions which give the absolute minimum of the action satisfy

$$\mathscr{D}_\mu \phi^i = 0, \qquad \partial V / \partial \phi^i = 0, \tag{2.5}$$

and, in a typical symmetry breaking situation, the potential $V(\boldsymbol{\phi})$ is chosen so that these equations admit solutions other than $\phi = 0$ which typically lie in the bottom of a 'valley' in V. For example, if $\mathrm{G} = \mathrm{SO}(n)$ and $V(\boldsymbol{\phi}) = (\boldsymbol{\phi} \cdot \boldsymbol{\phi} - a^2)(\boldsymbol{\phi} \cdot \boldsymbol{\phi} - b^2)$ $(i = 1 \ldots n)$ then the classical stable 'vacuum' configurations are $\boldsymbol{\phi} \cdot \boldsymbol{\phi} = a^2$ or $\boldsymbol{\phi} \cdot \boldsymbol{\phi} = b^2$.

In general, if $\boldsymbol{\phi}$ belongs to a real n-dimensional representation of G, the possible solutions of $\partial V / \partial \phi^i = 0$ lie on various submanifolds of $\mathbb{R}^n$ ($(n-1)$-spheres in the example above) and G acts transitively on each one. Each submanifold (G-orbit) has its own isotropy group H, and hence the corresponding vacuum Higgs–Kibble fields $\boldsymbol{\phi}$ take their values in the coset space G/H. This however is only a local picture, and globally $\boldsymbol{\phi}$ must follow the twists in the gauge structure present if the G-sector contains a non-vanishing topological charge. There then arises the possibility that, for topological reasons, such fields cannot be defined on all of M and consequently the spontaneous symmetry breaking from G to H is inhibited. In order to discuss this question adequately, it is necessary to be more precise about the mathematical status of the Higgs–Kibble fields.

2.2.

The language of fibre bundle theory is frequently employed in discussions of gauge theories, but it is all too often nothing but a language and the intrinsic power of the formalism is not invoked. However, the situation in our case is quite different, as the global topological properties of M play a vital part and it is precisely towards the topological structure that the non-trivial aspects of bundle theory are directed.

The starting point is a principal G-bundle P over M (Mayer 1977, Daniel and Viallet 1980) in which the Yang–Mills field is represented by a connection. A non-vanishing topological charge corresponds to P being non-trivial and is represented mathematically by the appropriate characteristic classes. If the Higgs–Kibble fields locally take their values in a vector space W, the correct mathematical picture is obtained by regarding them as cross sections of the associated vector bundle $P \times_{\mathrm{G}} W$ (for a review see Mayer (1981)). The symmetry breaking equation $\partial V / \partial \phi^i = 0$ assigns the range of $\boldsymbol{\phi}$ to a fixed G-orbit in the vector space W with isotropy group H, and hence these particular vacuum solutions correspond to cross sections of the associated fibre bundle $P \times_{\mathrm{G}} \mathrm{G/H}$. Now $P \times_{\mathrm{G}} \mathrm{G/H}$ is naturally isomorphic to P/H—the principal bundle P with the H-action factored out—and cross sections of this bundle are in one-to-one correspondence with reductions of the structure group from G to H (Kobayashi and Nomizu 1963). With each such reduction there is associated a principal H-bundle Q with an H-covariant embedding l of Q into P. This may be summarised in the map diagram

$$\begin{tikzcd}
\mathrm{H} \arrow[r] & Q \arrow[r, "\iota"] \arrow[d, "p_{\phi}"] & P \arrow[d, "p"] \\
& M \arrow[r, "\phi"] \arrow[dr, "\mathrm{id}"'] & P/\mathrm{H} \arrow[d, "q"] \\
& & M
\end{tikzcd} \tag{2.6}$$

where p and q are respectively the projection maps from P onto P/H and $P \times_{\mathrm{G}} \mathrm{G}/\mathrm{H} = P/\mathrm{H}$ onto M and are related to the projection map π of the original G-bundle $P \to M$ by $\pi = q \cdot p$. The bundle (Q, p_{ϕ}, M) is the pull back of $(P, p, P/\mathrm{H})$ with $Q = \{(x, e) \in M \times P \,|\, \phi(x) = p(e)\}$ and $p_{\phi}(x, e) := p(x)$.

The key observations are that, depending on the topological properties of M and the topological charge of P:

(i) Such reductions may or may not exist. In the latter case there is a topological obstruction to $\mathrm{G} \Rightarrow \mathrm{H}$.

(ii) If reduction is permitted, homotopically inequivalent Higgs–Kibble fields may lead to inequivalent H-bundles, i.e. H-sectors with different 'topological charges'.

The actual mathematical problem is to describe, in as simple a manner as possible, the relations between the topological charges and properties of M.

3. Characteristic classes and spontaneous symmetry breaking

3.1.

Universal bundle theory and characteristic classes are singularly appropriate for tackling this problem. The starting point is that, for any Lie group G, there exists a space $B\mathrm{G}$ (unique up to homotopy type) such that the set $B_{\mathrm{G}}(X)$ of isomorphism classes of principal G-bundles over any 'nice' space X (CW complex) is in one-to-one correspondence with the set $[X, B\mathrm{G}]$ of homotopy classes of maps from X into $B\mathrm{G}$ (Borel 1967, Milnor and Stasheff 1974, Switzer 1975). A map $f_{\xi}: X \to B\mathrm{G}$ which represents a particular bundle ξ is said to be a *classifying map.*

For certain groups G, $B\mathrm{G}$ may be represented by a well known space. For example $B\mathrm{U}1 \sim S^1$, $BZ_2 \sim \mathbb{R}P^{\infty}$ and $B\mathrm{U}1 \sim \mathbb{C}P^{\infty}$ where $\mathbb{R}P^{\infty}$ and $\mathbb{C}P^{\infty}$ are respectively infinite-dimensional real and complex projective spaces. Provided that one is prepared to limit the dimension of X, simple models for $B\mathrm{G}$ may also be exhibited for other groups. In this context $\mathrm{SU}(n)$ and $\mathrm{SO}(n)$ have complex and real Grassmann manifolds as classifying spaces and, if $\dim M \leq 4$, an example of $B\mathrm{SU}2$ is the four-sphere. These facts are discussed in the cited texts and in Avis and Isham (1979).

Universal characteristic classes for G are any non-vanishing cohomology elements of $B\mathrm{G}$. If f_{ξ} classifies ξ and $C \in \mathrm{H}^i(B\mathrm{G}; \pi)$, where π is an Abelian group, then $C(\xi) := f_{\xi}^*(C)$ is said to be a *characteristic class for* ξ. In the physics literature, the characteristic classes of a given G-bundle are usually known as the topological charges of the corresponding gauge sector. A feature of particular physical importance is that G-bundles over a manifold M can be completely and uniquely classified by certain sets of characteristic classes provided that $\dim M \leqslant 4$ (Dold and Whitney 1959, Avis and Isham 1979, Woodward 1980). For example $B_{\mathrm{U1}}(M) = \mathrm{H}^2(M; \mathrm{Z})$, $B_{\mathrm{SU}(n)}(M) = \mathrm{H}^4(M; \mathrm{Z})$ and $B_{\mathrm{U}(n)}(M) = \mathrm{H}^2(M; \mathrm{Z}) \oplus \mathrm{H}^4(M; \mathrm{Z})$ $(n \geqslant 2)$ where the two- and four-dimensional cohomology elements are known as the first and second Chern classes.

3.2.

When discussing the symmetry breaking problem, the first step is to realise that, if H is a subgroup of G with embedding $J : \mathrm{H} \to \mathrm{G}$, then any H-bundle η can be regarded as a G-bundle ξ. There is a universal map $BJ : B\mathrm{H} \to B\mathrm{G}$ such that the classifying map for ξ is $BJ \circ h$ where h classifies η. Conversely the structure group of a G-bundle ξ, with classifying map $f : M \to B\mathrm{G}$, can be reduced to H if (and only if) f can be factored through a map $h : M \to B\mathrm{H}$ such that $f = BJ \circ h$. Hence symmetry breaking from G to H is possible if and only if there exists a function g in the following diagram extension of (2.10):

$$\begin{tikzcd}
\mathrm{H} \arrow[r] & Q \arrow[r, "i"] \arrow[d, "p_\phi"] & P \arrow[d, "p"] & \\
& M \arrow[r, "\phi"] \arrow[dr, "\mathrm{id}"'] & P/\mathrm{H} \arrow[r, "\omega"] \arrow[d, "q"] & B\mathrm{H} \arrow[d, "BJ"] \\
& & M \arrow[ur, dashed, "h"'] \arrow[r, "f"'] & B\mathrm{G}
\end{tikzcd} \qquad (3.1)$$

The maps ω and $\omega \circ \phi = h$ classify the principal H-bundles $(P, p, P/\mathrm{H})$ and (Q, p_ϕ, M) respectively and h exists if and only if the Higgs–Kibble field ϕ exists.

Note that $\phi_1 \sim \phi_2$ implies $\omega \circ \phi_1 \sim \omega \circ \phi_2$ and hence homotopically equivalent Higgs–Kibble fields induce isomorphic H-bundles Q. On the other hand, if a pair of non-homotopic maps ϕ_1 and ϕ_2 lead to non-homotopic $\omega \circ \phi_1$ and $\omega \circ \phi_2$, then the Q-bundles will be inequivalent. This is precisely the $\mathrm{G} \overset{\rightarrow}{\rightarrow} \mathrm{H}$ phenomenon.

The aim of our work is to find necessary and sufficient conditions on the topological charges/characteristic classes of ξ for h to exist and hence symmetry breaking to occur. Clearly this is a viable proposition only because of the feasibility, mentioned above, of cohomologically classifying G-bundles when $\dim M \leqslant 4$. These conditions are closely connected with the relationships between the topological charges of ξ and η (the reduced H-bundle Q) that *necessarily* arise when $\mathrm{G} \Rightarrow \mathrm{H}$. These necessary relationships occur in answering the question: 'Given an H-bundle η, what are its G-characteristic classes when viewed as a G-bundle?' For example if $(\mathrm{G}, \mathrm{H}) = (\mathrm{SU2}, \mathrm{U1})$ the appropriate question is: 'What is the second Chern class $C_2(\xi)$ of the SU2-bundle ξ obtained from a U1 bundle η, classified by its first Chern class $C_1(\eta)$, when U1 is viewed as a subgroup of SU2?' We shall see in § 4 that the answer is

$$C_2(\xi) = -C_1(\eta) \cup C_1(\eta). \qquad (3.2)$$

(In contemplating such an equation it may be helpful to recall that in the Chern–Weil theory the real (rather than integral) cohomology classes are represented by differential forms and the cup product by the exterior product (Kobayashi and Nomizie 1969).)

Thus (3.2) is a necessary relation between $C_2(\xi)$ and $C_1(\eta)$ for SU2 $\Rightarrow$ U1. On the other hand, let us start with an SU2-bundle ξ classified by $C_2(\xi)$ in $H^4(M; Z)$ and look for sufficient conditions for SU2 $\Rightarrow$ U1. Suppose that there exists some α in $H^2(M; Z)$ such that

$$C_2(\xi) = -\alpha \cup \alpha. \tag{3.3}$$

Then α must be the first Chern class of *some* U1 bundle η and, if it is viewed as an SU2-bundle ξ', we know that $C_2(\xi') = -\alpha \cup \alpha$. However, SU2-bundles are uniquely classified by their second Chern classes and hence $\xi \approx \xi'$. Since, by construction, the structure group of ξ' can be reduced to U1, then so can the structure group of ξ, and hence (3.3) is a *sufficient* condition for SU2 $\Rightarrow$ U1. Furthermore, the first Chern class of the ensuing U1-bundle is precisely α.

Note that in general the topology of M will be such that an α satisfying (3.3) only exists for certain values of $C_2(\xi)$, and therefore symmetry breaking will occur only in those SU2-sectors, while in the rest it is obstructed. On the other hand, there may be several α satisfying (3.3) for a given $C_2(\xi)$, corresponding to the symmetry breaking of SU2 into different U1-sectors. For example, let $M = S^2 \times S^2$. Then $H^2(M; Z) = Z \oplus Z$ and $H^4(M; Z) = Z$ with generators γ_L, γ_R and δ satisfying

$$\delta = -\gamma_L \cup \gamma_R. \tag{3.4}$$

Any α in $H^2(M; Z)$ can be written as a sum $\alpha = s\gamma_L + r\gamma_R$ with integer coefficients s and r. Clearly $\alpha \cup \alpha = 2sr\gamma_L \cup \gamma_R$. On the other hand, $C_2(\xi) = p\delta$ for some integer p, and hence SU2 $\Rightarrow$ U1 only if the second Chern number p can be written as $p = 2sr$ for some s and r (cf Duff and Madore 1979).

For an arbitrary pair of Lie groups G and H we can proceed as follows. Let $\{C_1 \ldots C_r\}$ and $\{d_1 \ldots d_s\}$ be sets of universal classes which, when $\dim M \leqslant 4$, uniquely classify G- and H-bundles respectively. There may be relations within a set of the form

$$\begin{aligned} &R_k(C_1 \ldots C_r) = 0, \qquad && k = 1 \ldots K, \\ &S_l(d_1 \ldots d_s) = 0, \qquad && l = 1 \ldots L, \end{aligned} \tag{3.5}$$

where cup products, coefficient reductions and occasionally other cohomology operations may occur. For example SO(n)-bundles ($n \geqslant 5$) are classified by the set of elements (W_2, p) in $H^2(M; Z_2) \oplus H^4(M; Z)$ (the second Stieffel–Whitney and Pontryagin classes) satisfying $p \bmod 2 = W_2 \cup W_2$. Note that other characteristic classes may exist but they are functionally related to those appearing already. For example, in an SO(n)-bundle, W_3 is determined by W_2.

If $h: M \to BH$ represents an H-bundle η then the characteristic classes $\{d_j(\eta) = h^* d_j \in H^{n_j}(M; \pi'_j)\}$ satisfy

$$S_l(d_1(\eta) \ldots d_s(\eta)) = 0, \qquad l = 1 \ldots L, \tag{3.6}$$

and the set $\{d_1 \ldots d_s\}$ must be chosen so that any set of cohomology classes $\{\alpha_j \in H^{n_j}(M; \pi'_j)\}$ satisfying $S_l(\alpha_1 \ldots \alpha_s) = 0$, $l = 1 \ldots L$, are the cohomology classes of *some* H-bundle (this is possible if $\dim M \leqslant 4$). Similar remarks apply to the set $\{C_1 \ldots C_r\}$.

Now, let η be such a bundle and consider the characteristic classes $C_i(\xi) = (BJ \circ h)^* C_i \in \mathrm{H}^{n_i}(M; \pi_i)$ of η viewed as a G-bundle ξ. There will be relations

$$C_i(\xi) = P_i(d_1(\eta) \ldots d_s(\eta)), \qquad i = 1 \ldots r, \tag{3.7}$$

which identically satisfy

$$R_k(C_1(\xi) \ldots C_r(\xi)) = 0, \qquad k = 1 \ldots K. \tag{3.8}$$

Equations (3.7) are *necessary* conditions for $\mathrm{G} \Rightarrow \mathrm{H}$. On the other hand, arguing as in the $\mathrm{SU2} \Rightarrow \mathrm{U1}$ example, a *sufficient* condition for the symmetry breaking of a given G-bundle ξ is the existence of $\{\alpha_j \in \mathrm{H}^{\eta_j}(M; \pi'_j), j = 1 \ldots s\}$ satisfying

$$\begin{aligned} &S_l(\alpha_1 \ldots \alpha_s) = 0, \qquad l = 1 \ldots L, \\ &C_i(\xi) = P_i(\alpha_1 \ldots \alpha_s) \qquad i = 1 \ldots r. \end{aligned} \tag{3.9}$$

The characteristic classes of the reduced bundle η are simply $d_j(\eta) = \alpha_j$ and the existence, for fixed ξ, of different sets of $\alpha_1 \ldots \alpha_s$ satisfying (3.9) leads to G H.

Now $C_i(\xi) = (BJ \circ h)^*(C_i) = h^* BJ^*(C_i)$ and $d_j(\eta) = h^* d_j$. Then (3.7) gives, for every h,

$$h^*(P_i(d_1 \ldots d_s) - BJ^*(C_i)) = 0, \qquad i = 1 \ldots r, \tag{3.10}$$

which implies

$$P_i(d_1 \ldots d_s) = BJ^*(C_i), \qquad i = 1 \ldots r, \tag{3.11}$$

since the elements in $\mathrm{H}^{\eta_j}(M; \pi'_j)$ uniquely specify H-bundles. Thus the polynomials P_i, which give the solution to the whole problem of relating the existence of spontaneous symmetry breaking to the presence of topological charge, may be found by studying relations between the universal characteristic classes. For many pairs of groups (G, H) these relations are readily available in the mathematical literature, and it is usually relatively straightforward to compute the rest from scratch.

4. Specific examples

4.1.

The simplest example is $\mathrm{G} = \mathrm{SU}(n)$ and $\mathrm{H} = \mathbb{1}$ (the trivial subgroup). In this case $B\mathrm{H}$ is a single point, and since an $\mathrm{SU}(n)$-bundle on a four-manifold is classified by its second Chern class, equation (3.11) becomes $J^*(C_2) = 0$. Hence symmetry breaking from $\mathrm{SU}(n)$ to $\mathbb{1}$ occurs only when the topological charge is zero. Similarly $\mathrm{SO}(n) \Rightarrow \mathbb{1}$ if and only if $p(\xi) = W_2(\xi) = 0$.

In general, one can exploit the fact that the map $BJ: B\mathrm{H} \to B\mathrm{G}$ is actually a fibre map with fibre the coset space G/H (Borel 1967). Information on the action of BJ^* on the cohomology groups of $B\mathrm{G}$ can be abstracted from the Serre short exact cohomology sequence of the bundle $\mathrm{G/H} \to B\mathrm{H} \xrightarrow{BJ} B\mathrm{G}$:

$$0 \to \mathrm{H}^1(B\mathrm{G}; \pi) \xrightarrow{BJ^*} \mathrm{H}^1(B\mathrm{H}; \pi) \to \mathrm{H}^1(\mathrm{G/H}; \pi) \xrightarrow{\tau} \mathrm{H}^2(B\mathrm{G}; \pi) \longrightarrow. \tag{4.1}$$

The group beyond $\mathrm{H}^2(B\mathrm{G}; \pi)$ to which this sequence can be extended depends on the number of lower-order cohomology groups of $B\mathrm{G}$ and G/H that vanish (Borel 1967, Spanier 1966). In the following symmetry breaking calculations it is easy to check that we are within the range for which Serre's sequence is exact.

4.2.

Next consider SU3 $\Rightarrow$ SU2 with SU3/SU2 = S^5 and so $H^i(\mathrm{SU3/SU2}; Z) = 0$ for $1 \leq i \leq 4$. Then (4.1) with Z-coefficients gives

$$0 \to H^4(B\mathrm{SU3}; Z) \xrightarrow{BJ^*} H^4(B\mathrm{SU2}; Z) \to 0. \tag{4.2}$$

However $H^4(B\mathrm{SU2}; Z) = H^4(B\mathrm{SU3}; Z) = Z$ with the second Chern class as generator in both cases. Thus, in an obvious notation, BJ^*C_2 (SU3) = $\pm C_2$(SU2). The sign ambiguity is conventionally resolved by *defining* the SU3-class to be the same as SU2, and thus (3.11) reads

$$BJ^*C_2(\mathrm{SU3}) = C_2(\mathrm{SU2}). \tag{4.3}$$

The analysis of § 3.2 shows that SU3 breaks to SU2 without obstruction and into a unique sector. Similarly SU4/SU3 = S^7 implies SU4 $\Rightarrow$ SU3, and in general we get the chain

$$\Rightarrow \mathrm{SU}(n) \Rightarrow \cdots \Rightarrow \mathrm{SU4} \Rightarrow \mathrm{SU3} \Rightarrow \mathrm{SU2} \tag{4.4}$$

with no obstructions or splitting at any stage.

Since $\mathrm{SO}(n+1)/\mathrm{SO}(n) = S^n$ the same technique is applicable (if $n \geq 5$), giving

$$\Rightarrow \mathrm{SO}(n) \Rightarrow \cdots \Rightarrow \mathrm{SO7} \Rightarrow \mathrm{SO6} \Rightarrow \mathrm{SO5} \tag{4.5}$$

with

$$BJ^*p(\mathrm{SO}(n+1)) = p(\mathrm{SO}(n)), \qquad BJ^*W_2(\mathrm{SO}(n+1)) = W_2(\mathrm{SO}(n)). \tag{4.6}$$

The situation when $n = 4$ is complicated by the appearance of an extra characteristic class (the Euler class e) in $H^4(M; Z)$ in the classification of SO4-bundles. Again one finds $BJ^*p(\mathrm{SO5}) = p(\mathrm{SO4})$ and $BJ^*W_2(\mathrm{SO5}) = W_2(\mathrm{SO4})$ and SO5 $\Rightarrow$ SO4 without obstruction. However, the Euler class is left unspecified and can therefore label inequivalent SO4-sectors leading to the splitting SO5 $\rightrightarrows$ SO4.

When G = SO4 and H = SO3 the Z-Serre sequence gives

$$0 \to Z \xrightarrow{\tau} H^4(B\mathrm{SO4}; Z) \xrightarrow{BJ^*} H^4(B\mathrm{SO3}; Z) \tag{4.7}$$

and, as before, $BJ^*p(\mathrm{SO4}) = p(\mathrm{SO3})$ and $BJ^*W_2(\mathrm{SO4}) = W_2(\mathrm{SO3})$. However the Euler class e is defined so that $\tau(\iota) = e$ (ι is an appropriately chosen generator of Z) and hence, by exactness, $BJ^*(e) = 0$. Thus a necessary and sufficient condition for SO4 $\Rightarrow$ SO3 is $e(\xi) = 0$ and there is no splitting.

The SO3 $\Rightarrow$ SO2 = U1 case was studied many years ago (albeit not in the context of spontaneous symmetry breaking!) by Massey (1958) and Dold and Whitney (1959) who found

$$BJ^*p(\mathrm{SO3}) = -C_1 \cup C_1, \qquad BJ^*W_2(\mathrm{SO}_3) = C_1 \bmod 2. \tag{4.8}$$

Thus a necessary condition for SO3 $\Rightarrow$ SO2 is $p(\xi) = -C_1(\eta) \cup C_1(\eta)$ and $W_2(\xi) = C_1(\eta) \bmod 2$, whilst a sufficient condition is the existence of α in $H^2(M; Z)$ such that $p(\xi) = -\alpha \cup \alpha$ and $W_2(\xi) = \alpha \bmod 2$. As in the SU2 $\Rightarrow$ U1 example, equation (4.8) leads to a blocking of $SO3 \Rightarrow SO2$ from certain SO3-sectors and an $SO3 \rightrightarrows SO2$ splitting.

4.3.

An interesting example is $G = SU(n+1)$, $H = U(n)$ with $SU(n+1)/U(n) = \mathbb{C}P^n$. The embedding of $U(n)$ into $SU(n+1)$ is $J : U \rightsquigarrow \left(\begin{array}{c|c} U & 0 \\ \hline 0 & \det U^{-1} \end{array}\right)$ which can be factored as

$$\begin{tikzcd} U(n) \arrow[r, "\Delta"] \arrow[drrr, "J"'] & U(n)\times U(n) \arrow[r, "1\times\det^{-1}"] & U(n)\times U(1) \arrow[r, "k"] & U(n+1) \\ & & & SU(n+1) \arrow[u, "l"'] \end{tikzcd} \tag{4.9}$$

where $\Delta(U) = (U, U)$ and k and l are the usual embeddings. The maps 'det' and l arise in the fibre bundle $SU(n) \xrightarrow{l} U(n) \xrightarrow{\det} U(1)$ and associated fibration (Borel 1967) $BSU(n) \xrightarrow{Bl} BU(n) \xrightarrow{B\det} BU1$. Using the Serre sequence and $H^i(BSU(n); Z) = 0$, $1 \leqslant i \leqslant 3$, we obtain

$$\begin{array}{ccc} 0 \to H^2(BU1; Z) & \xrightarrow{B\det^*} & H^2(BU(n); Z) \to 0, \\ Z(C_1) & & Z(C_1) \end{array} \tag{4.10}$$

and

$$\begin{array}{ccccc} 0 \to H^4(BU1; Z) & \xrightarrow{B\det^*} & H^4(BU(n); Z) & \xrightarrow{Bl^*} & H^4(BSU(n); Z) \to 0, \\ Z(C_1 \cup C_1) & & Z(C_1 \cup C_1) \oplus Z(C_2) & & Z(C_2) \end{array} \tag{4.11}$$

where the groups and generators have been written below the exact sequences.

Now (4.10) implies $B\det^* C_1(U1) = C_1(U(n))$ and hence $B\det^*(C_1(U1) \cup C_1(U1)) = C_1(U(n)) \cup C_1(U(n))$ which, on using the exactness of (4.11), gives $Bl^*(C_1 \cup C_1) = 0$ and $Bl^* C_2(U(n)) = C_2(SU(n))$. Applying the last result to (4.9) with the 'B' operation applied, we obtain

$$\begin{aligned} BJ^* C_2(SU(n+1)) &= BJ^* Bl^* C_2(U(n+1)) \\ &= B\Delta^* B(1 \times \det^{-1})^* Bk^* C_2(U(n+1)). \end{aligned} \tag{4.12}$$

Now the Whitney formula (Borel 1967, Milnor and Stasheff 1974) gives $Bk^* C_2(U(n+1)) = C_2(U(n)) \times 1 + C_1(U(n)) \times C_1(U1)$ and $\det^{-1}$ is det followed by $U1 \to U1$, $\lambda \rightsquigarrow \lambda^{-1}$ which takes $C_1(U1)$ into $-C_1(U1)$. Hence (4.12) gives the appropriate form of (3.11):

$$BJ^* C_2(SU(n+1)) = C_2(U(n)) - C_1(U(n)) \cup C_1(U(n)). \tag{4.13}$$

Thus a necessary condition for an $SU(n+1)$-bundle ξ to break to a $U(n)$-bundle η is $C_2(\xi) = C_2(\eta) - C_1(\eta) \cup C_1(\eta)$, whilst a sufficient condition is the existence of $\alpha \in H^4(M; Z)$ and $\beta \in H^2(M; Z)$ such that

$$C_2(\xi) = \alpha - \beta \cup \beta. \tag{4.14}$$

Clearly this imposes no restrictions if $n > 1$ (so that $C_2(U(n)) \neq 0$) and $SU(n+1) \Rightarrow U(n)$ is always allowed with a splitting $SU(n+1) \rightrightarrows U(n)$. Note that when $n = 1$ we get $C_2(U(1)) = 0$ and hence derive the $SU2 \Rightarrow U1$ results cited earlier.

4.4.

Finally, let us discuss some of the groups that are currently employed in unified and grand unified theories. It is most important to establish the global form of the group. Thus the Salam–Weinberg $SU2_L \times U1$ is globally U2, as is clear from the typical field transformations under $(A_L, e^{i\theta}) \in SU2_L \times U1$,

$$\begin{pmatrix}\phi^+\\ \phi^0\end{pmatrix} \to A_L e^{i\theta}\begin{pmatrix}\phi^+\\ \phi^0\end{pmatrix}, \qquad \begin{pmatrix}\nu_e\\ e^-\end{pmatrix}_L \to A_L e^{-i\theta}\begin{pmatrix}\nu_e\\ e^-\end{pmatrix}_L \qquad \text{etc} \tag{4.15}$$

where (ϕ^+, ϕ^0) is the Higgs pair. The Z_2 subgroup

$$\left\{\left(\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}, 1\right), \left(\begin{pmatrix}-1 & 0\\ 0 & -1\end{pmatrix}, -1\right)\right\}$$

acts trivially, so the effective group of transformations is $SU2_L \times U1/Z_2 = U2$. The $U1_{em}$ subgroup is embedded in U2 as $J(e^{i\phi}) = \begin{pmatrix}e^{2i\phi} & 0\\ 0 & 1\end{pmatrix}$ and in $SU2_L \times U1$ as

$$J(e^{i\phi}) = \left(\begin{pmatrix}e^{i\phi} & 0\\ 0 & e^{-i\phi}\end{pmatrix}, e^{i\phi}\right).$$

The topological properties of symmetry breaking $G \Rightarrow U1_{em}$ depend very much on whether G is chosen to be U2 or $SU2_L \times U1$, which in turn is dictated by the particle multiplets in the theory.

The $U(n) \Rightarrow U(n-1) \Rightarrow \cdots$ chain is very similar to the $SU(n)$ series discussed in § 4.1. In general, $BJ^*C_2(U(n)) = C_2(U(n-1))$ and $BJ^*C_1(U(n)) = C_1(U(n-1))$. In particular, $BJ^*C_2(U(2)) = 0$ and $BJ^*C_1(U(2)) = C_1(U(1))$. Hence $U(2) \Rightarrow U(1)$ if and only if the 'instanton number' (i.e. second Chern class) of the U(2)-sector vanishes. On the other hand, using an analysis similar to that employed in § 4.3, the $SU2_L \times U1 \Rightarrow U1_{em}$ result is

$$BJ^*C_2(SU2_L) = -C_1(U1_{em} \cup U1_{em}), \qquad BJ^*C_1(U1) = C_1(U1_{em}), \tag{4.16}$$

which gives both blocking and splitting of the symmetry breaking.

4.5.

A very popular grand unified theory is based on the group SU5 which breaks at 10^{15} GeV to $(SU2_L \times U1 \times SU3_c)/Z_6$ and at 100 GeV to U3. The cyclic group Z_6 is the kernel of the homomorphism from $SU2_L \times U1 \times SU3_c$ into SU5 given by

$$(A, e^{i\theta}, C) \rightsquigarrow \left(\begin{array}{c|c} C\,e^{-2i\theta} & 0\\ \hline 0 & A\,e^{3i\theta}\end{array}\right)$$

and has the generator

$$\left(\begin{pmatrix}-1 & 0\\ 0 & -1\end{pmatrix}, e^{i\pi/3}, e^{i4\pi/3}\begin{pmatrix}1 & & \\ & 1 & \\ & & 1\end{pmatrix}\right)$$

(see Scott 1980). The U3 subgroup is locally the product of $U1_{em}$ and the colour group

$SU3_c$, and is embedded in $(SU2_L \times U1 \times SU3_c)/Z_6$ by

$$i(U) = \left[\begin{pmatrix} \det U^{-1/2} & 0 \\ 0 & \det U^{-1/2} \end{pmatrix}, \det U^{-1/6}, \frac{U}{\det U^{1/3}}\right]_{Z_6} \tag{4.17}$$

where the ambiguous roots are chosen so that $\det U^{1/2} \det U^{1/6} \det U^{1/3} = \det U$.

Let H denote $(SU2_L \times U1 \times SU3_c)/Z_6$ with k as the subgroup embedding of H in SU5. Using $l: H \to U2 \times U3$ defined by $l[A, e^{i\theta}, C]_{Z_6} := (A\, e^{3i\theta}, C\, e^{-2i\theta})$ and the obvious embedding m of SU5 in U5, we may construct the commutative diagrams:

$$\begin{array}{ccc} H & \xrightarrow{l} & U2 \times U3 \\ \downarrow J & & \downarrow k \\ SU5 & \xrightarrow{m} & U5 \end{array} \qquad \begin{array}{ccc} BH & \xrightarrow{Bl} & B(U2 \times U3) \\ \downarrow BJ & & \downarrow Bk \\ BSU5 & \xrightarrow{Bm} & BU5 \end{array} \tag{4.18}$$

H is a normal subgroup of $U2 \times U3$ and is the kernel of the map $t: U2 \times U3 \to U1$ defined by $t(A, C) = \det A \det C$. This induces (Borel 1967) the fibration

$$U1 \to BH \xrightarrow{Bl} B(U2 \times U3) \tag{4.19}$$

which may be employed as follows to compute the universal characteristic classes of H.

The Gysin sequence with Z-coefficients (Spanier 1966) of (4.19) gives the long exact sequence (with $B \equiv B(U2 \times U3)$)

$$\begin{array}{l} 0 \to \underset{Z}{H^0(B)} \xrightarrow{\cup e} \underset{Z \oplus Z}{H^2(B)} \xrightarrow{Bl^*} H^2(BH) \to \underset{O}{H^1(B)} \xrightarrow{\cup e} H^3(B) \to H^3(BH) \to \underset{Z}{H^2(B)} \\ \qquad \xrightarrow{\cup e} \underset{\oplus_{i=1}^5 Z}{H^4(B)} \xrightarrow{Bl^*} H^4(BH) \to \underset{O}{H^3(B)} \to \cdots \end{array} \tag{4.20}$$

where the generators of $H^2(BU2 \times BU3; Z)$ and $H^4(BU2 \times BU3; Z)$ are respectively $(C_1 \times 1, 1 \times C_1)$ and $(C_2 \times 1, C_1^2 \times 1, C_1 \times C_1, 1 \times C_1^2, 1 \times C_2)$ where $C_1^2 \equiv C_1 \cup C_1$. Note that (4.19) also implies that $\pi_1(BH) = \pi_0(H) = 0$ and $\pi_2(BH) = \pi_1(H) = Z$. Then $H^2(BH; Z) \approx Z$ and clearly Bl^* cannot vanish on both $C_1 \times 1$ and $1 \times C_1$. However, with reference to (4.18), $Bm^*(C_1) = 0$ and hence $0 = Bl^* Bk^* C_1 = Bl^*(C_1 \times 1 + 1 \times C_1)$ by Whitney duality and commutativity of (4.18). Thus $Bl^*(C_1 \times 1) = -Bl^*(1 \times C_1) = \delta$, say and then, by exactness of (4.20), $e = C_1 \times 1 + 1 \times C_1$.

It now follows that in $H^2(BU2 \times BU3; Z)$ we have $(C_1 \times 1)Ue = C_1^2 \times 1 + C_1 \times C_1$ and similarly $(1 \times C_1) \cup e = 1 \times C_1^2 + C_1 \times C_1$. Then Ue acting on $H^2(BU2 \times BU3; Z)$ has kernel $\{0\}$ and hence, by exactness of (4.20), $H^3(BH; Z) = 0$. Using exactness once again, we have $Bl^*(C_1^2 \times 1 + C_1 \times C_1) = Bl^*(1 \times C_1^2 + C_1 \times C_1) = 0$ and hence $Bl^*(C_1^2 \times 1) = Bl^*(1 \times C_1^2) = -Bl^*(C_1 \times C_1) = \delta \cup \delta$ and so the relevant cohomology groups of $BH \equiv B(SU2_L \times U1 \times SU3_c)/Z_6$ are

$$H^2(BH; Z) = Z \qquad \text{with generator } \delta = Bl^*(C_1 \times 1)$$

and

$$H^4(BH; Z) = Z \oplus Z \oplus Z$$

with generators $C_{2L} := Bl^*(C_2 \times 1)$, $C_{2c} := Bl^*(1 \times C_2)$ and $\delta \cup \delta$. (4.21)

Now $BJ^*C_2(\mathrm{SU5}) = BJ^*Bm^*C_2(\mathrm{U5}) = Bl^*Bk^*C_2(\mathrm{U5}) = Bl^*\{C_2(\mathrm{U2}) \times 1 + 1 \times C_2(\mathrm{U3}) + C_1(\mathrm{U2}) \times C_1(\mathrm{U3})\}$ by Whitney duality. Hence the important map (3.11) is

$$BJ^*C_2(\mathrm{SU5}) = C_{2\mathrm{L}} + C_{2\mathrm{c}} - \delta \cup \delta \tag{4.22}$$

with a necessary condition for $\mathrm{SU5} \Rightarrow (\mathrm{SU2_L} \times \mathrm{U1} \times \mathrm{SU3_c})|\mathrm{Z}_6$ being

$$C_2(\xi) = C_{2\mathrm{L}}(\eta) + C_{2\mathrm{c}}(\eta) - \delta(\eta) \cup \delta(\eta) \tag{4.23}$$

and a corresponding sufficient condition leading to splitting but no inhibition. Note how the initial topological charge can 'flow' into different channels and be taken up by $C_{2\mathrm{L}}$, $C_{2\mathrm{c}}$ and $\delta \cup \delta$ in many different ways.

4.6.

The second stage of symmetry breaking is $(\mathrm{SU2_L} \times \mathrm{U1} \times \mathrm{SU3_c})/\mathrm{Z}_6 \Rightarrow \mathrm{U3}$, which can be studied via the diagrams

$$\begin{tikzcd} \mathrm{U3} \arrow[r, "i"] \arrow[dr, "j"'] & (\mathrm{SU2_L} \times \mathrm{U1} \times \mathrm{SU3_c})/\mathrm{Z}_6 \arrow[d, "l"] \\ & \mathrm{U2} \times \mathrm{U3} \end{tikzcd} \qquad \begin{tikzcd} B\mathrm{U3} \arrow[r, "Bi"] \arrow[dr, "Bj"'] & B(\mathrm{SU2_L} \times \mathrm{U1} \times \mathrm{SU3_c})/\mathrm{Z}_6 \arrow[d, "Bl"] \\ & B\mathrm{U2} \times B\mathrm{U3} \end{tikzcd} \tag{4.24}$$

where

$$j(\mathrm{U}) = \left(\begin{pmatrix} 1 & 0 \\ 0 & \det \mathrm{U}^{-1} \end{pmatrix}, \mathrm{U}\right)$$

which factorises as

$$\mathrm{U3} \xrightarrow{\Delta} \mathrm{U3} \times \mathrm{U3} \xrightarrow{\det^{-1} \times 1} \mathrm{U1} \times \mathrm{U3} \xrightarrow{r \times 1} \mathrm{U2} \times \mathrm{U3}$$

with

$$r(\mathrm{e}^{\mathrm{i}\phi}) := \begin{pmatrix} 1 & 0 \\ 0 & \mathrm{e}^{\mathrm{i}\phi} \end{pmatrix}.$$

We are interested in $Bi^*C_{2\mathrm{L}} = Bi^*Bl^*(C_2 \times 1) = Bj^*(C_2 \times 1)$, $Bi^*(C_{2\mathrm{c}}) = Bi^*Bl^*(1 \times C_2) = Bj^*(1 \times C_2)$ and $Bi^*(\delta \cup \delta) = -Bi^*Bl^*(C_1 \times C_1) = -Bj^*(C_1 \times C_1)$. From § 4.3 we have $B \det^* C_1 = C_1$ and clearly $(r \times 1)^*(C_2 \times 1) = 0$ and $(r \times 1)^*(1 \times C_2) = C_2$. Hence

$$Bi^*C_{2\mathrm{L}} = 0, \qquad Bi^*C_{2\mathrm{c}} = C_2(\mathrm{U3}), \qquad Bi^*(\delta \cup \delta) = C_1(\mathrm{U3}) \cup C_1(\mathrm{U3}). \tag{4.25}$$

Thus a necessary and sufficient condition for symmetry breaking of $(\mathrm{SU3_L} \times \mathrm{U1} \times \mathrm{SU3_c})/\mathrm{Z}_6$ to U3 is $C_{2\mathrm{L}} = 0$. Note that in § 4.5 we saw via (4.23) that a given SU5 sector could break into a $(\mathrm{SU3_L} \times \mathrm{U1} \times \mathrm{SU3_c})/\mathrm{Z}_6$ sector for which $C_{2\mathrm{L}} \neq 0$. In this case the final breaking to the physical subgroup U3 would be inhibited.

4.7.

Another popular grand unified group is SO10 breaking down to SU5. In general, we could study $\mathrm{SO}(2n) \Rightarrow \mathrm{SU}(n)$, and the effect of BJ^* on the cohomology of $B\mathrm{SO}(2n)$ is well known (e.g. Borel 1967) to be

$$BJ^*p(\mathrm{SO}(2n)) = C_2(\mathrm{SU}(n)), \qquad BJ^*W_2(\mathrm{SO}(2n)) = 0. \tag{4.26}$$

Hence the only obstruction to symmetry breaking is a non-vanishing $W_2(\xi)$ which is equivalent to the global non-existence of SO$(2n)$ spinors.

5. Conclusions

We have seen that it is possible to express the necessary and sufficient conditions for spontaneous symmetry breaking G$\Rightarrow$H in terms of certain functional relationships between the topological charges of the initial and final gauge sectors. The nature of these calculations illustrates once again the usefulness of the fibre bundle formalism of Yang–Mills theories when augmented with the cohomology theory necessary to give a complete description of the characteristic classes.

In a functional integral quantisation scheme there will presumably be a summation of all initial and final sectors including those pairs in which symmetry breaking is blocked. It is clearly important to understand the nature of the stable field configurations in this case, and to see whether the value of their action is such that they are effectively suppressed in the functional integral. In Isham (1981b) this problem is discussed briefly, and it is shown that for values of the scalar coupling constant below a critical value the stable solution is $\phi = 0$, so there is complete symmetry restoration. It is not known what happens above this critical value, but a heuristic argument suggests that a stable solution will only exist if there exists a minimal area, three-dimensional submanifold of space–time on which ϕ vanishes. This problem is being actively pursued and the results will be published in a later paper.

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